MATH 105 101 Midterm 2 Sample 4 Solutions

1. (20 marks)

(a) (5 marks) Determine the intervals on which the following function is increasing or decreasing:

$$F(x) = \int_{x^2}^{0} (t+9)\,dt.$$

Solution: In order to find the intervals on which the function is increasing or decreasing, we want to find critical points. So, first we need the derivative using the Fundamental Theorem of Calculus Part I, and chain rule, we get:

$$\begin{aligned} F(x) &= \int_{x^2}^{0} (t+9)\,dt = -\int_{0}^{x^2} (t+9)\,dt \\ \Rightarrow \frac{dF}{dx} &= -(x^2+9)(2x), \end{aligned}$$

which is 0 for $x = 0$. Testing the sign of the derivatives on each subintervals, we get:

- On $(-\infty, 0)$: $\frac{dF}{dx} > 0$ (for example, $\frac{dF}{dx}\mid_{x=-1} = 20$).
- On $(0, \infty)$: $\frac{dF}{dx} < 0$ (for example, $\frac{dF}{dx}\mid_{x=1} = -20$).

Thus, the function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

(b) (5 marks) Use the Trapezoidal Rule to approximate

$$\int_{-1}^{1} \arccos(x)\,dx$$

with $n = 4$ subintervals. **Simplify the answer.**

Solution: We have $a = -1$, $b = 1$, $n = 4$, and $f(x) = \arccos(x)$. So, $\Delta x = \frac{b-a}{n} = 1/2$. There are 5 grid-points using the formula $x_k = a + k\Delta x$:

$$x_0 = -1, \quad x_1 = -1/2, \quad x_2 = 0, \quad x_3 = 1/2, \quad x_4 = 1.$$

Using Trapezoidal Rule, we get:

$$\begin{aligned}
T_4 &= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) \\
&= \frac{1/2}{2}(\arccos(-1) + 2\arccos(-1/2) + 2\arccos(0) + 2\arccos(1/2) + \arccos(1)) \\
&= \frac{1}{4}(\pi + 2\left(\frac{2\pi}{3}\right) + 2\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{3}\right) + 0) \\
&= \frac{1}{4}(4\pi) = \pi.
\end{aligned}$$

(c) (5 marks) Compute the Left Riemann sum for $f(x) = \frac{x}{2}$ on the interval $[-2, 6]$ using $n = 4$ equal subintervals. **Simplify the answer.**

Solution: We have $a = -2$, $b = 6$, and $n = 4$. So, $\Delta x = \frac{b-a}{n} = 2$. For Left Riemann sum, we have:

$$x_k^* = a + (k-1)\Delta x = -2 + (k-1)2 = -4 + 2k.$$

So, $f(x_k^*) = \frac{-4+2k}{2} = -2 + k$. Thus, The Left Riemann sum for $f(x) = \frac{x}{2}$ on $[-2, 6]$ using $n = 4$ equal subintervals is:

$$\begin{aligned}
&f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x \\
&= 2(-2+1) + 2(-2+2) + 2(-2+3) + 2(-2+4) \\
&= -2 + 0 + 2 + 4 = 4.
\end{aligned}$$

(d) (5 marks) Find the definite integral

$$\int_0^2 \frac{x}{x^2-4}\,dx.$$

Solution: Note that $\frac{x}{x^2-4}$ is undefined at $x = 2$, so this is an improper integral, and:

$$\int_0^2 \frac{x}{x^2-4}\,dx = \lim_{b\to 2^-}\int_0^b \frac{x}{x^2-4}\,dx.$$

Use substitution with $u = x^2 - 4$, we get $du = 2x\,dx$. So,

$$\int \frac{x}{x^2-4}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-4| + C.$$

So,

$$\lim_{b\to 2^-}\int_0^b \frac{x}{x^2-4}\,dx = \lim_{b\to 2^-}\frac{1}{2}\ln|x^2-4|\,\Big|_0^b = \lim_{b\to 2^-}\frac{1}{2}\ln|b^2-4| - \frac{1}{2}\ln 4.$$

As $b\to 2^-$, then $b^2-4\to 0^-$, and $|b^2-4|\to 0^+$, which means $\ln|b^2-4|\to -\infty$. Thus, the limit does not exist, and the improper integral diverges.

2. (10 marks) Evaluate the definite integral:

$$\int_0^{\frac{\pi}{6}} \frac{2x\sin(x)}{\cos^3(x)}\,dx.$$

Solution: First note that $\frac{2x\sin(x)}{\cos^3(x)}$ is continuous on $[0,\pi/6]$, so it is not improper. To integrate this, we want to first use integration by parts (since we have a product of trigonometric functions and x). Let $u=2x$ and $dv=\frac{\sin(x)}{\cos^3(x)}\,dx$. Then, $du=2dx$.

To find v, we first find $\int\frac{\sin(x)}{\cos^3(x)}\,dx$, so we need to use a simple substitution with $t=\cos(x)$, then $dt=-\sin(x)\,dx$. So,

$$\int\frac{\sin(x)}{\cos^3(x)}\,dx = \int\frac{-1}{t^3}\,dt = \frac{t^{-2}}{2}+C = \frac{1}{2\cos^2(x)}+C = \frac{1}{2}\sec^2(x)+C.$$

So, $v=\frac{1}{2}\sec^2(x)$. Then,

$$\begin{aligned}\int_0^{\frac{\pi}{6}}\frac{2x\sin(x)}{\cos^3(x)}\,dx &= x\sec^2(x)\,\Big|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\sec^2(x)\,dx\\ &= \left(\frac{\pi}{6}\left(\frac{2}{\sqrt{3}}\right)^2-0\right)-\tan(x)\,\Big|_0^{\frac{\pi}{6}}\\ &= \frac{2\pi}{9}-\left(\left(\frac{1}{\sqrt{3}}\right)-0\right)=\frac{2\pi}{9}-\frac{1}{\sqrt{3}}.\end{aligned}$$

3. (10 marks) Evaluate the indefinite integral:

$$\int\frac{x}{\sqrt{x^2-2x+5}}\,dx.$$

Solution: First, we complete the square, and get $\sqrt{x^2-2x+5} = \sqrt{(x-1)^2+4}$. So, first we use a simple substitution $t = x-1$, with $dt = dx$ to obtain:

$$\int \frac{x}{\sqrt{x^2-2x+5}}\,dx = \int \frac{t+1}{\sqrt{t^2+4}}\,dt = \int \frac{t}{\sqrt{t^2+4}}\,dt + \int \frac{1}{\sqrt{t^2+4}}\,dt.$$

For $\int \frac{t}{\sqrt{t^2+4}}\,dt$, a simple substitution with $s = t^2-4$ and $ds = 2t\,dt$ yields:

$$\int \frac{t}{\sqrt{t^2+4}}\,dt = \int \frac{1}{2\sqrt{s}}\,ds = s^{1/2} + C = \sqrt{t^2-4} + C.$$

For $\int \frac{1}{\sqrt{t^2+4}}\,dt$, we use trigonometric substitution with $t = 2\tan\theta$, and $dt = 2\sec^2\theta d\theta$. Furthermore, $\sqrt{t^2+4} = 2\sec\theta$. So,

$$\int \frac{1}{\sqrt{t^2+4}}\,dt = \int \frac{1}{2\sec\theta}(2\sec^2\theta\,d\theta) = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C.$$

Note that if $t = 2\tan\theta$, then $\sec\theta = \frac{\sqrt{t^2+4}}{2}$, and:

$$\ln|\sec\theta+\tan\theta| + C = \ln|\frac{\sqrt{t^2+4}}{2} + \frac{t}{2}| + C.$$

Hence,

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-2x+5}}\,dx &= \sqrt{t^2-4} + \ln|\frac{\sqrt{t^2+4}}{2} + \frac{t}{2}| + C \\ &= \sqrt{(x-1)^2-4} + \ln|\frac{\sqrt{(x-1)^2-4}}{2} + \frac{x-1}{2}| + C.\end{aligned}$$

4. (10 marks) Solve the initial value problem:

$$\frac{dy}{dt}e^{2y}(t^2+1) - t = 0, \qquad y(0) = 0.$$

You may leave the answer in its implicit form.

Solution: We have:

$$\frac{dy}{dt}e^{2y}(t^2+1) - t = 0 \Leftrightarrow \frac{dy}{dt}e^{2y}(t^2+1) = t \Rightarrow e^{2y}\,dy = \frac{t}{t^2+1}\,dt.$$

Next, we want to integrate each side with respect to the respective variables. The left hand side yields:

$$\int e^{2y}\,dy = \frac{e^{2y}}{2} + C.$$

For the integral $\int \frac{t}{t^2+1}\,dt$, we use substitution with $x = t^2+1$ and $dx = 2t\,dt$. Then,

$$\int \frac{t}{t^2+1}\,dt = \int \frac{1}{2u}\,dx = \frac{\ln|u|}{2} + C = \frac{\ln|t^2+1|}{2} + C.$$

So, $\frac{e^{2y}}{2} = \frac{\ln|t^2+1|}{2} + C$. To find C, we use the initial condition $y(0) = 0$, and get:

$$\frac{e^0}{2} = \frac{\ln|1|}{2} + C \Rightarrow C = 1/2.$$

So, the solution to the initial value problem in its implicit form is:

$$\frac{e^{2y}}{2} = \frac{\ln|t^2+1|}{2} + 1/2.$$